A Frequency-Dependent Basis Function Applied to Microstrip

Charles Hechtman, Henry Zmuda, and David Gabbay

Abstract —**Spheroidal wave functions and the spectral-domain method are used to compute the effective dielectric constant for microstrip. A single-term expansion for the vector current density provides excellent results over a broad spectrum (1–100 GHz). Numerical results compare favorably with other commonly used techniques.**

I. Introduction

Microstrip is an important planar transmission structure and has many applications (see, for example, [1]). The most popular quasi-static (approximate) microstrip solutions were found by Wheeler [2], [3]. Later Denlinger [4] gave a frequency-dependent (dispersive) microstrip solution. This solution centers on a Fourier transform approach using an approximate current distribution. Subsequently, Itoh and Mittra [5] enhanced the technique by using the spectral-domain method (SDM) [6]. This combines Denlinger's Fourier formulation with the moment method [7] for determining the propagation constant and the current distribution simultaneously. With the SDM, the current is sought in terms of a truncated series of aperture limited basis functions.

Initially Itoh and Mittra [5] used a pulse and triangle as basis functions to simulate the current distribution. Later, Uwano and Itoh [8] realized that it would take many terms to simulate the edge singularities and suggested sinusoidal functions with appropriate edge conditions as an orthogonal basis set. (Note that apertures with infinitely sharp edges require that the current density vector follow specific behavior near the edge. This is known as the edge condition.) Moreover, other basis functions have been suggested for similar aperture limited type problems, e.g., Chebyshev polynomials [9] and Legendre polynomials [10]. It is known that an appropriate choice of basis function results in an accurate solution with a small number of expansion terms [11]. In addition, a candidate for a single-term current representation must be frequency dependent [12] and therefore excludes the above basis functions. Such considerations are important for MMIC CAD applications, especially where radiation effects must be taken into account. Rhodes, in his scholarly work on synthesis of planar antenna sources [13], introduced spheroidal wave functions (SWF) for aperture radiation problems. In his work he exploited the double orthogonal properties of SWF's for far-field synthesis. In this present communication we explore the ability of the SWF to change shape as a function of a parameter, e.g., frequency, while maintaining orthogonality, completeness, edge condition, and aperture limit.

In the next section we introduce the SDM equations and provide a brief overview of the SWF. We then study the effective dielectric constant as a function of frequency for several commonly used basis functions.

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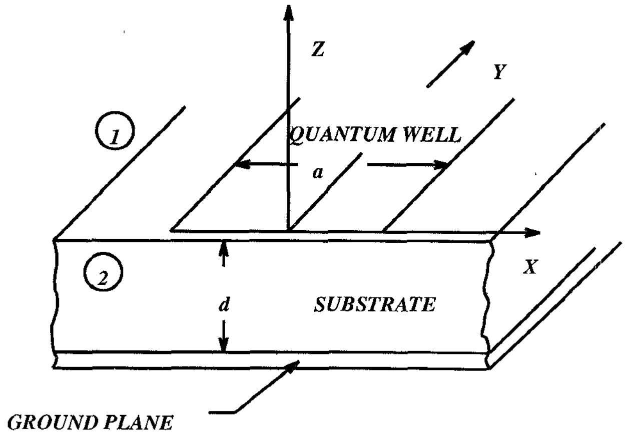

Fig. 1. Configuration of microstrip transmission line structure.

II. Formulation

Fig. 1 shows the microstrip transmission line with an appropriate coordinate system. The structure extends infinitely in the y direction. The strip, centered at the origin, is of width a, infinitesimally thin, and perfectly conducting. A perfectly conducting ground plane is located a distance d below the strip as shown. Region 1 is of infinite extent above the strip, the permittivities of dielectric regions 1 and 2 are ϵ_1 and ϵ_2, respectively, and permeability μ_0 is assumed throughout. Field quantities vary in time as $e^{j\omega t}$.

Using the vector potential formulation [6] of the SDM, the electric field, $\tilde{\boldsymbol{E}}$, is related to the current, $\tilde{\boldsymbol{J}}$, confined on the strip by

$$\tilde{\boldsymbol{E}}(\alpha,\beta)=\tilde{\boldsymbol{G}}^{-1}(\alpha,\beta)\tilde{\boldsymbol{J}}(\alpha,\beta) \quad (1)$$

where $\tilde{\boldsymbol{G}}^{-1}$ is the matrix inverse of $\tilde{\boldsymbol{G}}$,

$$\tilde{\boldsymbol{G}}(\alpha,\beta)=\frac{-1}{\omega\mu_0}\cdot\begin{bmatrix}\dfrac{k_1^2-\beta^2}{\gamma_1}-j\dfrac{k_2^2-\beta^2}{\gamma_2}\cot\gamma_2 d & \alpha\beta\left(\dfrac{1}{\gamma_1}-j\dfrac{1}{\gamma_2}\cot\gamma_2 d\right)\\ \alpha\beta\left(\dfrac{1}{\gamma_1}-j\dfrac{1}{\gamma_2}\cot\gamma_2 d\right) & \dfrac{k_1^2-\alpha^2}{\gamma_1}-j\dfrac{k_2^2-\alpha^2}{\gamma_2}\cot\gamma_2 d\end{bmatrix} \quad (2)$$

and

$$\tilde{\boldsymbol{E}}=\begin{bmatrix}\tilde{E}_x\\ \tilde{E}_y\end{bmatrix} \qquad \tilde{\boldsymbol{J}}=\begin{bmatrix}\tilde{J}_x\\ \tilde{J}_y\end{bmatrix}. \quad (3)$$

The Fourier transforms of the current density vector, $\boldsymbol{J}$, and the electric field, $\boldsymbol{E}$, are given by

$$\tilde{\boldsymbol{J}}(\alpha,\beta)=\iint_{-\infty}^{\infty}\boldsymbol{J}(x,y)e^{-j(\alpha x+\beta y)}\,dx\,dy \quad (4)$$

$$\tilde{\boldsymbol{E}}(\alpha,\beta)=\iint_{-\infty}^{\infty}\boldsymbol{E}(x,y)e^{-j(\alpha x+\beta y)}\,dx\,dy \quad (5)$$

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where

$$\gamma_i^2 = k_i^2 - \alpha^2 - \beta^2 \qquad k_i^2 = \epsilon_i k_0^2$$
$$k_0^2 = \omega^2 \mu_0 \epsilon_0, \qquad i = 1,2. \tag{6}$$

It is required to find the current density, $\tilde{J}$, and the propagation constant, β, that force $E_x = E_y = 0$ on the metal strip. The solution is found by expanding the current density in terms of aperture limited basis functions [8] (the current must of course be zero off the strip):

$$\tilde{J}_x = \sum_{m=1}^{N} c_m \tilde{J}_{xm} \qquad \tilde{J}_y = \sum_{m=1}^{M} d_m \tilde{J}_{ym} \tag{7}$$

where c_m and d_m are constants to be determined. The expansions are substituted into (1) and by Parseval's theorem and by requiring that the structure be infinite in the y direction, the following matrix equation is formed:

$$\int_{-\infty}^{\infty} \left[\tilde{J}_{xk} \tilde{G}_{11}^{-1} \sum_{m=1}^{N} c_m \tilde{J}_{xm} + \tilde{J}_{xk} \tilde{G}_{21}^{-1} \sum_{m=1}^{M} d_m \tilde{J}_{ym} \right] d\alpha = 0, \qquad k = 1,2,\cdots,N \tag{8}$$

$$\int_{-\infty}^{\infty} \left[\tilde{J}_{yk} \tilde{G}_{12}^{-1} \sum_{m=1}^{N} c_m \tilde{J}_{xm} + \tilde{J}_{yk} \tilde{G}_{22}^{-1} \sum_{m=1}^{M} d_m \tilde{J}_{ym} \right] d\alpha = 0, \qquad k = 1,2,\cdots,M. \tag{9}$$

In Itoh and Mittra's SDM microstrip solution the first terms ($M = N = 1$) of the current distribution are a pulse for the y component,

$$J_y = \begin{cases} 1, & |x| \leq a/2 \\ 0, & |x| > a/2 \end{cases} \tag{10}$$

and a truncated square wave for the x component,

$$J_x = \begin{cases} -1, & -a/2 \leq x \leq 0 \\ 1, & 0 < x \leq a/2 \\ 0, & |x| > a/2. \end{cases} \tag{11}$$

Additional terms may be added with triangle functions. The y and x component of $\boldsymbol{J}$ can be expressed as in Itoh's book [8]. He suggests

$$J_y = \frac{\cos[2(n-1)\pi x/a]}{\sqrt{1-(2x/a)^2}}, \qquad n = 1,2,\cdots \tag{12}$$

$$J_x = \frac{\sin[2n\pi x/a]}{\sqrt{1-(2x/a)^2}}, \qquad n = 1,2,\cdots. \tag{13}$$

The second set of basis functions possesses the edge conditions for the strip and, therefore, converges more rapidly than the first.

Shi *et al.* [12] showed the change in the current distribution as a function of frequency. The basis functions described above are independent of frequency and thus rely totally on a proper weighting of several functions to reflect the change in frequency.

In Rhodes's work on planar antenna synthesis he uses properties of SWF's to synthesize antenna radiation patterns. For such radiation problems, he shows that the SWF's are "natural functions" for aperture limited planar antennas. Although this paper deals only with the fundamental microstrip mode, which does not radiate, the SWF still serves as an excellent basis function. Moreover, while maintaining orthogonality, aperture limit, and edge conditions, these SWF's evolve to provide an approximation to the current variation with frequency. The SWF as described by Rhodes in [13, appendix III] is defined by a linear integral equation. It is modified and expressed as

$$\nu_{\tau,n}(c)\psi_{\tau,n}(c,\alpha/c) = \int_{-a/2}^{a/2} e^{-j\alpha x}\left(1-(2x/d)^2\right)^{\tau} \psi_{\tau,n}(c,x)\,dt, \qquad \tau > -1 \tag{14}$$

where $\psi_{\tau,n}(c,x)$ is the spheroidal wave function. The proportionality constant $\nu_{\tau,n}(c)$ is the characteristic number (i.e., eigenvalue) and c is a parameter that effects the distribution of the wave function. Note that (14) is essentially the Fourier transform of the product of a weighting function and the spheroidal wave function over the aperture ($-a/2$ to $a/2$), where α serves as the variable in the Fourier transform domain. The left-hand side of (14) is again the spheroidal wave function in the Fourier transform domain scaled by $1/c$. The SWF's are an entire function of α. They are real for real α, they have exactly n zeros within the interval $(-a/2, a/2)$, they are even or odd functions of α according to whether n even or odd, and they are orthogonal and complete on the intervals $(-a/2, a/2)$ and $(-\infty, \infty)$

The following edge conditions are satisfied [13] for both y and x components of the current:

$$J_{yn} = d_n\left(1-(2x/d)^2\right)^{-1/2} \psi_{-1/2,n}(c,x), \qquad \tau = -1/2 \tag{15}$$

and

$$J_{xn} = c_n\left(1-(2x/d)^2\right)^{1/2} \psi_{1/2,n}(c,x), \qquad \tau = 1/2 \tag{16}$$

where $\psi_{-1/2,n}(c,x) = \mathrm{Se}_n(c,x)$ and

$$\psi_{1/2,n}(c,x) = So_{n+1}(c,x)/(1-2x/a)^{1/2}$$

with Se_n and So_{n+1} being periodic even and odd Mathieu functions. This is consistent with the edge conditions in (12) and (13).

III. Numerical Results

We examine a microstrip structure with dimensions and dielectric values equivalent to the examples presented by Denlinger and Itoh. We compute the effective dielectric constant for several basis functions as a function of frequency. For all cases we compute an effective dielectric constant using a two-term SWF expansion per component ($M = N = 2$) in matrix equations (8) and (9). Owing to the symmetry of the problem we require only even terms for the y component and only odd terms for the x component. The current thus takes the following form:

$$J_{y1} = d_1\left(1-(2x/d)^2\right)^{-1/2} \psi_{-1/2,0}(c,x) \tag{17}$$

$$J_{y2} = d_2\left(1-(2x/d)^2\right)^{-1/2} \psi_{-1/2,2}(c,x) \tag{18}$$

where c is determined by trial and error to be $1.8k_0 a/2$. Similarly,

$$J_{x1} = c_1\left(1-(2x/d)^2\right)^{1/2} \psi_{1/2,1}(c,x) \tag{19}$$

$$J_{x2} = c_2\left(1-(2x/d)^2\right)^{1/2} \psi_{1/2,3}(c,x) \tag{20}$$

where c is determined by trial and error to be $1/(1+k_0 a/2)$.

Figs. 2–5 show the effective dielectric constant, ϵ_{eff}, as a function of frequency for microstrip consisting of a strip 3.17 mm wide, 3.07 mm above the ground plane, and having a

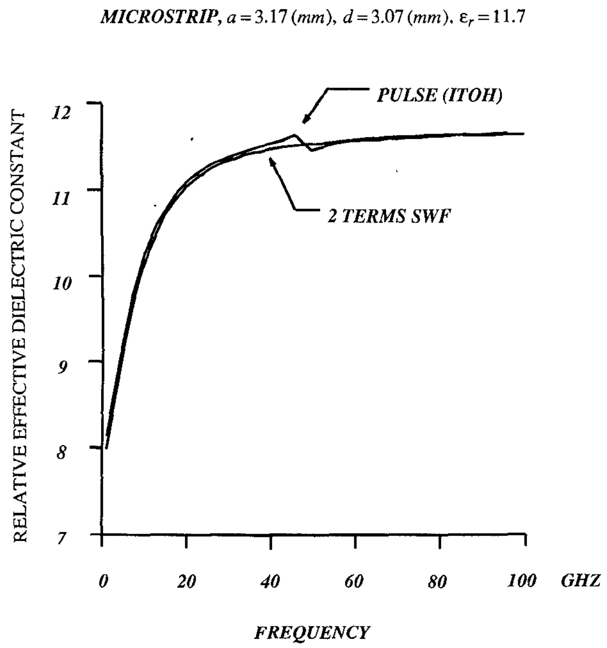

Fig. 2. Variation of effective dielectric constant with frequency (pulse).

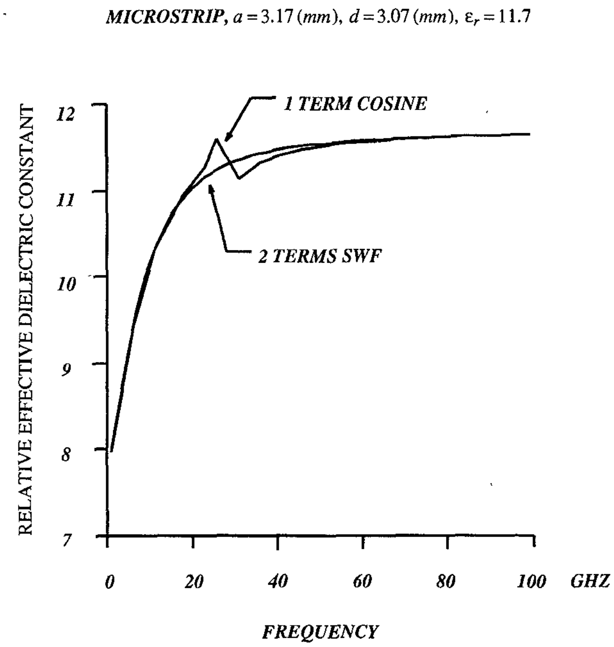

Fig. 3. Variation of effective dielectric constant with frequency (1 cosine).

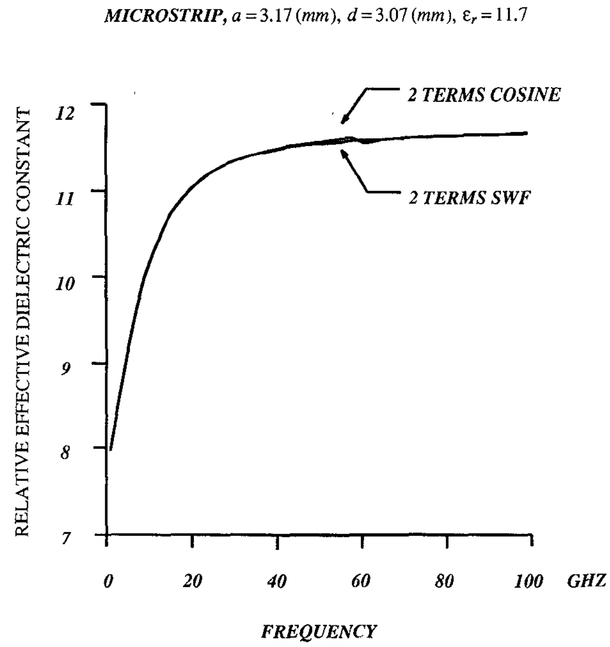

Fig. 4. Variation of effective dielectric constant with frequency (2 cosine).

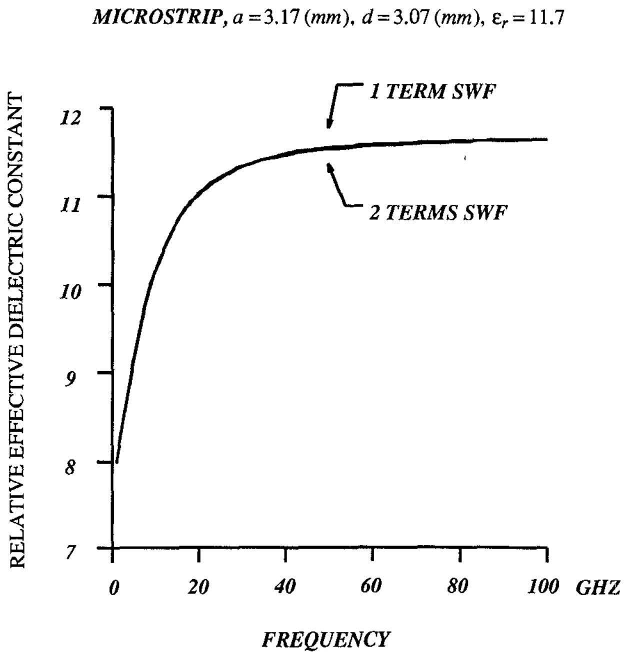

Fig. 5. Variation of effective dielectric constant with frequency (SWF).

relative dielectric constant $\epsilon_r = 11.7$. Fig. 2 compares the pulse and square wave basis function of (10) and (11) for ($M = N = 1$) with two terms of the SWF, (17)–(20), for ($M = N = 2$). Fig. 3 compares one term of the cosine and sine basis function of (12) and (13) for ($M = N = 1$) against the SWF ($M = N = 2$). Fig. 4 compares two terms of the cosine and sine basis function ($M = N = 2$) against the SWF ($M = N = 2$). Finally, Fig. 5 compares one term of the SWF ($M = N = 1$) against the SWF ($M = N = 2$). It is evident that a single term of the pulse (Fig. 2) or cosine and sine (Fig. 3) basis function does not follow the base line. However, two terms of the cosine function (Fig. 4) is a reasonable fit except at 50–65 GHz, where there is a significant departure. With one term of the SWF (Fig. 5), the two curves do not exhibit any major departure over the entire frequency range.

IV. Discussion

The SDM relies on a good choice of basis function to achieve rapid convergence. Rapid convergence will occur when a basis function is an excellent approximation to the actual eigenfunction. The basis functions that appear in the literature are excellent single-term approximations in the vicinity of a single frequency, namely dc. Unfortunately, the current distribution changes with frequency. Therefore, these basis functions have rapid convergence only in a limited region. In contrast, the first even and odd SWF's approximate the current distribution accurately for the fundamental mode of microstrip over a broad frequency range. Therefore, the SWF's are excellent basis functions when rapid convergence is required over a broad frequency spectrum.

To the best of the authors' knowledge, this is the first time SWF's have been applied to a guided microstrip problem. It is believed that the SWF's are ideally suited for aperture limited type problems that in general include radiation and guided waves.

Future work will center on the use of SWF's as basis functions for higher order modes and more complicated structures. In addition, a determination of the parameter c from basic principles would be desirable.

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